ESTIMATING THE SINE FUNCTION THROUGH CUBIC AND QUADRATIC FUNCTIONS.

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ABSTRACT:

When I was preparing for IIT-JEE I used to come across various sine values apart from the standard values which are to be memorised, but mathematics isn't about memorising, it's about learning. I always wondered there must be some formula/equation whose outputs are the values corresponding to those of sine wave after all sine wave is periodic and within 0 to π/2 it forms a curve which must be satisfied by at least a polynomial.

My idea is to find that there can be a cubic/quadratic equation of the form $ax^3 + bx^2 + cx + d = 0, \{a, b, c, d \in \mathbb{R}\}$ or $ax^2 + bx + c = 0, \{a, b, c, \in \mathbb{R}\}$ but their output will be the sine values.

QUESTION/ PROPOSAL:

Can there be a curve formed using basic tools of mathematics which is fit for the first 90 values of the sine wave and provide a reasonable approximation for the sine values?

PREVIOUS WORK ON IT:

1. Back in ancient times, long before calculus, a seventh-century Indian mathematician called Bhaskara I derived a simple approximation for the sine wave. This formula is given in his treatise titled Mahabhaskariya. It is not known how Bhaskara 1 derived this approximation. This approximation also has errors starting from third or fourth decimal places. The formula is given as:

$$\boxed{Sin\ (x) \approx \frac{16x(\pi - x)}{5\pi^2 - 4x(\pi - x)} (x\ in\ radians)}$$

2. In Calculus, Taylor Series is also used in approximation of the sine values for the angles close to zero.

$$Sin\ (x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \ldots\ \ (x\ in\ radians)$$

   So, for the values of x close to zero, sin (x) can be approximated as:

$$\boxed{Sin\ (x) \approx x - \frac{x^3}{3!}\ (x\ in\ radians)}$$

CUBIC APPROXIMATE FOR THE SINE WAVE:

APPROACH:

Now the general form of the cubic equation is written as: $ax^3 + bx^2 + cx + d$.

What I am planning to do is use random angles and their sine values in pair of 4 which should be in AP so as to ease the calculation. Some of the angles which I have used here are as follows:

{30,45,60,90}, {10,20,30,40}, {20,40,60,80}, {25,45,65,85}, {15,35,55,75}. Using these values to satisfy the general equation of the cubic and get the required variables a, b, c, d as follows (as an example):

$$0.342 = 8000a + 400b + 20c + d \qquad \text{... (1)}$$
$$0.642 = 64000a + 1600b + 40c + d \qquad \text{... (2)}$$
$$0.866 = 216000a + 3600b + 60c + d \qquad \text{... (3)}$$
$$0.984 = 512000a + 6400b + 80c + d \qquad \text{... (4)}$$

This is the way how I proceed for each set of 4 angles in AP. I got different values of these variables from each of these set of angles.

I got the nearest approximation to the sine wave with the following set of the variables.

$$a = -6.25 \times 10^{-7} \qquad b = -2 \times 10^{-5} \qquad c = +0.01795 \qquad d = 0$$

Now out of these values "a" and "c" can be approximated as:

$$\boxed{a \approx -6 \times 10^{-7} \qquad b = -2 \times 10^{-5} \qquad c \approx +0.018 \qquad d = 0}$$

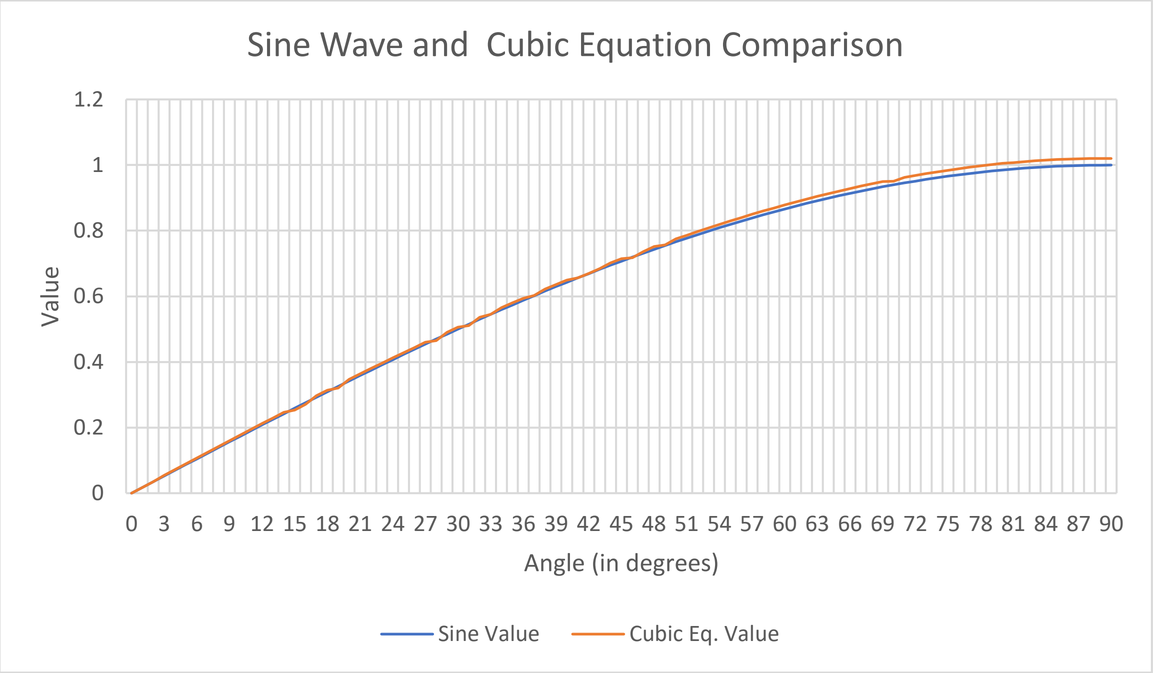
Each of these values of the variable is taken from a different set of angles, for example, the value of 'a' is taken from {20,40,60,80}, the value of 'd' is taken from {0,30,60,90} and so on:

Hence the cubic formed will be:

$$\boxed{y = -6 \times 10^{-7}x^3 - 2 \times 10^{-5}x^2 + 0.018x} \qquad \text{... (5)}$$

TESTING THE CUBIC FORMED:

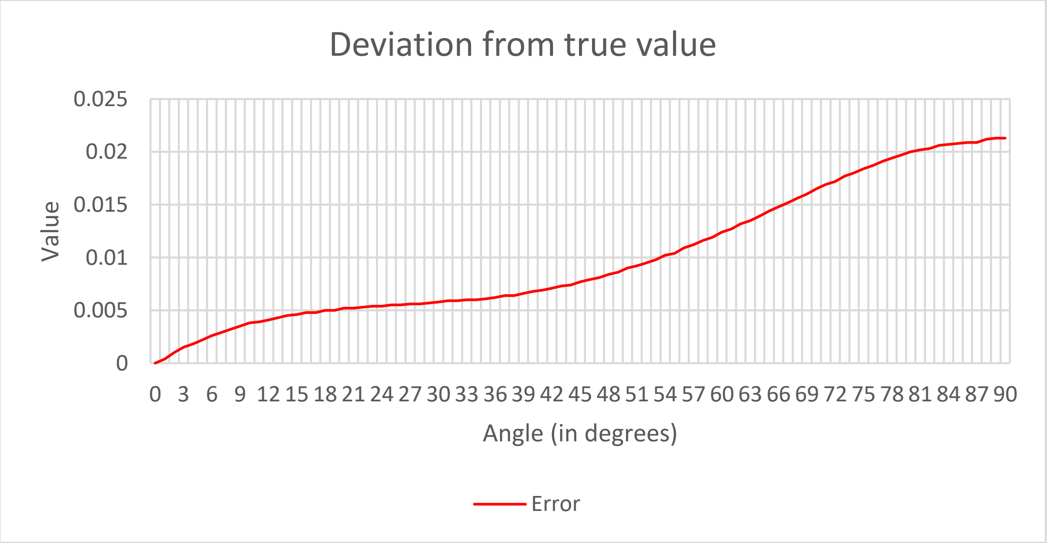
If I graph the actual values of the sine function (up to 4 decimal places) and the values obtained by cubic equation ($y = -6 \times 10^{-7}x^3 - 2 \times 10^{-5}x^2 + 0.018x$) (up to 4 decimal places) then the graph will look something like this:



It can be seen that the cubic equation follows the near approximation of the sine function till around 47 degrees. From there onwards the deviation is much higher.

ERROR IN CUBIC EQUATION

If we calculate the error in cubic equation concerning original sine function then:



Here, the deviation is huge compared to its actual value which doesn't seem to be a good approximate.

SHORTCOMINGS OF CUBIC EQUATION:

1. The amount of error/deviation is quite large as far as higher values are concerned.
2. This method is quite lengthy and also time-consuming from a calculation point of view (by calculations I mean getting a cube of a number). (For this reason, higher degree polynomials are not preferred)

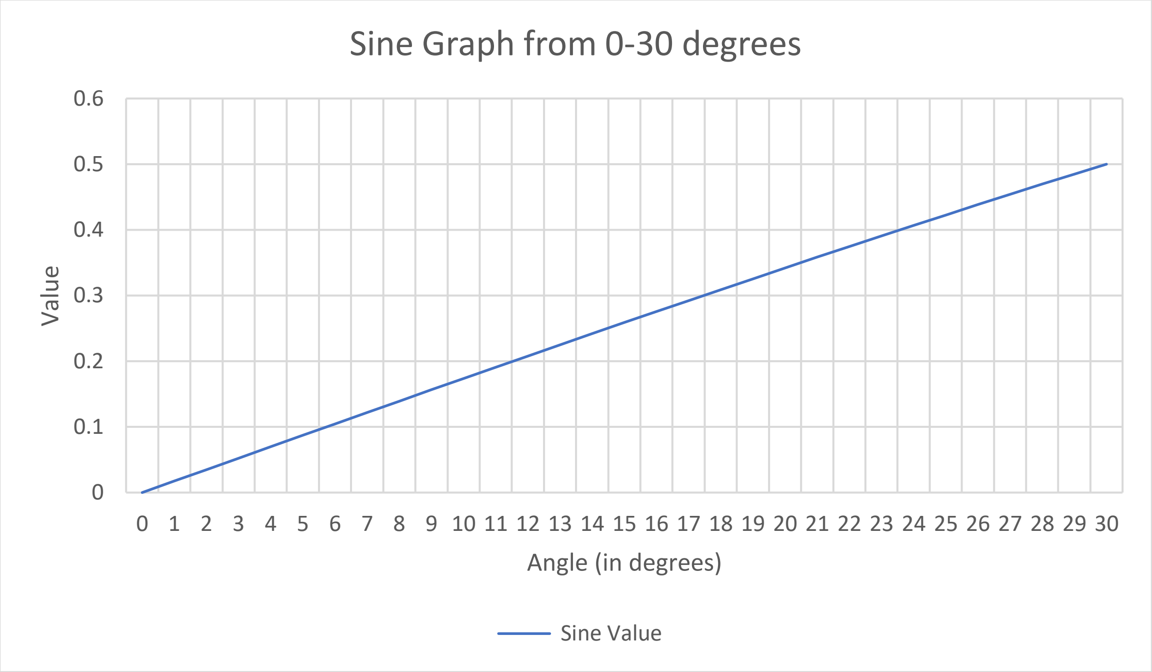
SOLUTION FOR THE SHORTCOMINGS:

To overcome the above drawback, the possible solution is to form a quadratic equation with the least possible chances of error.

THE QUADRATIC APPROXIMATE FOR THE SINE WAVE:

To calculate the quadratic approximate, I have divided the range (0-90) into three sub-ranges from (0-30), (31-60), (61-90) and I will take each of these sub-ranges and then combine all three to get the final quadratic equation.

1. Range from (0-30)



Let the quadratic equation for the given set of angles be of the form: $ax^2 + bx + c = 0$. Again, the same process is followed as above, I have tried various combination of triplets of angles, (5,15,25); (0,15,30); (7,17,27); (9,18,27) etc. out of all those triplets, (9, 18, 27) provides the best value of constants, 'a' and 'b'.

The equations used to get the values of 'a' and 'b' are:

$$0.1564 = 81a + 9b + c \qquad \dots (6)$$
$$0.3090 = 324a + 18b + c \qquad \dots (7)$$
$$0.4540 = 729a + 27b + c \qquad \dots (8)$$

On solving these equations, we get the value of 'a' as $-4.6 \times 10^{-5} \approx -4 \times 10^{-5}$ and the value of 'b' as $+0.0182 \approx +0.018$

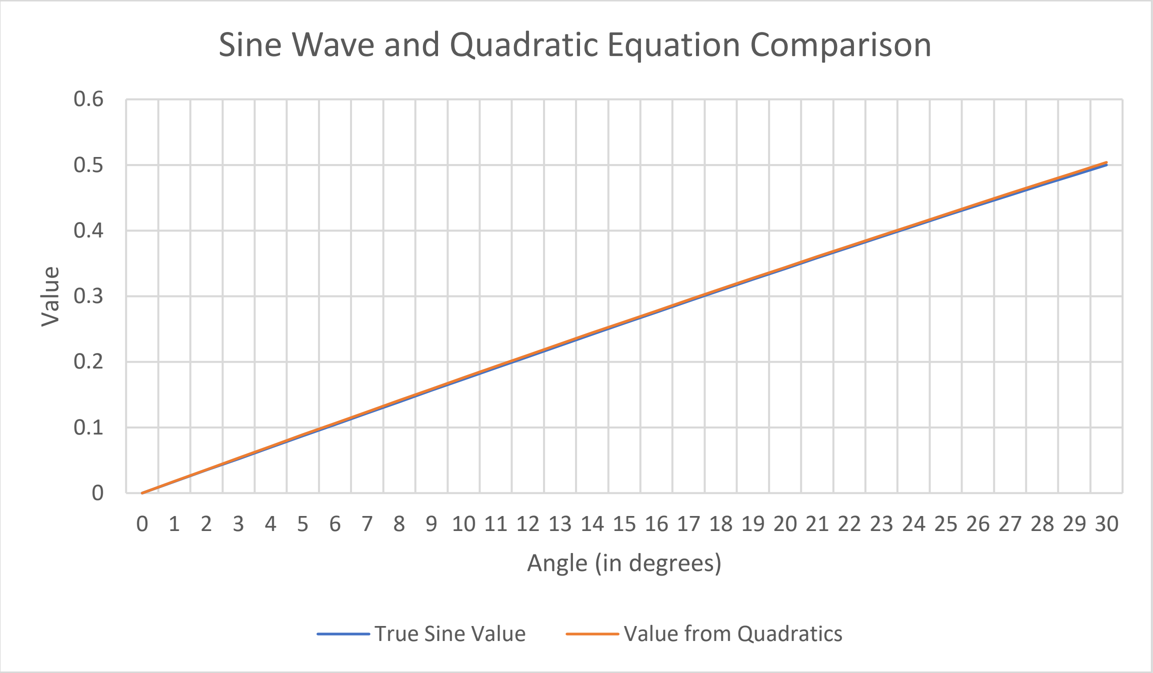
Now to calculate the value of 'c' I will follow the different approach as the value of 'c' tells how much the graph has to be shifted upwards or downwards along the y-axis.

To begin the calculation of ‘c’ let us assume that $c = 0$ and form the remaining quadratic after that calculating its error/deviation and then using those deviations to get the required value of ‘c’ with the least possible error.

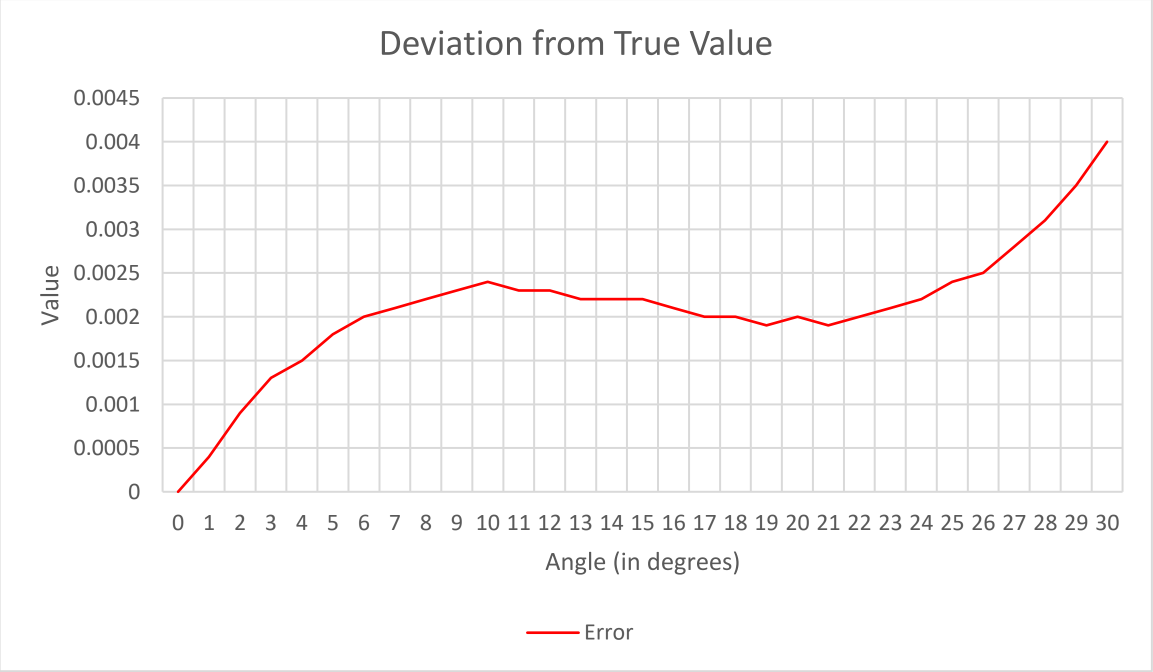
So, let the quadratic equation be: $y = -4 \times 10^{-5}x^2 + 0.018x + 0$.

GRAPH AND IT’S DEVIATION FROM THE ACTUAL VALUE:

If we graph the actual values of the sine function (up to 4 decimal places) and the values obtained by quadratic equation ($y = -4 \times 10^{-5}x^2 + 0.018x + 0$) (up to 4 decimal places) then the graph will look something like this.



As we see that both the graphs almost overlap each other but there are errors in both the values. If we graph the errors then,



Now since the maximum error occurs at '30' which is '0.0040' and all of these errors are positive so the value of 'c' will have the negative sign and to get the value of 'c' we will use the method of averages as the quadratic equation cannot be formed with the given graph of errors and if we go for a higher degree of polynomial then main motive to find the quadratic equation will be affected.

So, by the method of averages, we get: $\frac{\Sigma\, Errors}{30} = \frac{0.0645}{30} \approx 0.0020.$

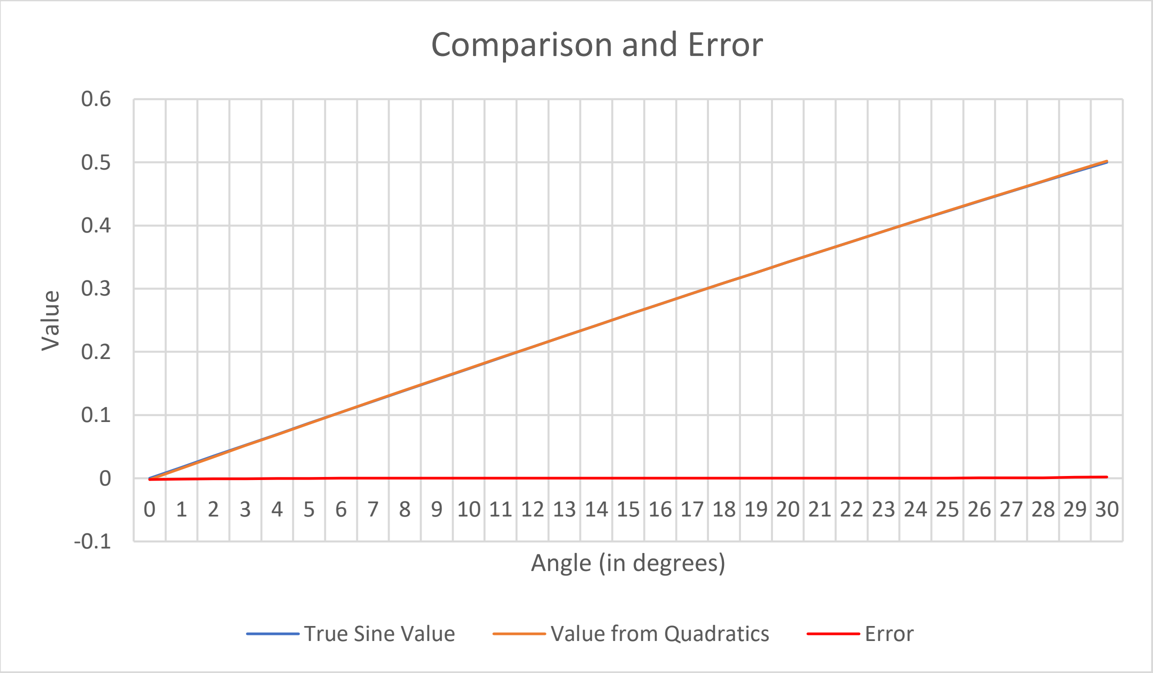
Hence the value of the average of the errors is 0.0021 which is our required value of 'c' that must be subtracted as a constant in the quadratic equation. This might change the value of quadratic at x=0, but on the contrary, it will closely estimate the other values for the given angles and the error will be significantly reduced.

Hence the final equation for the first sub-range (0-30) will be:

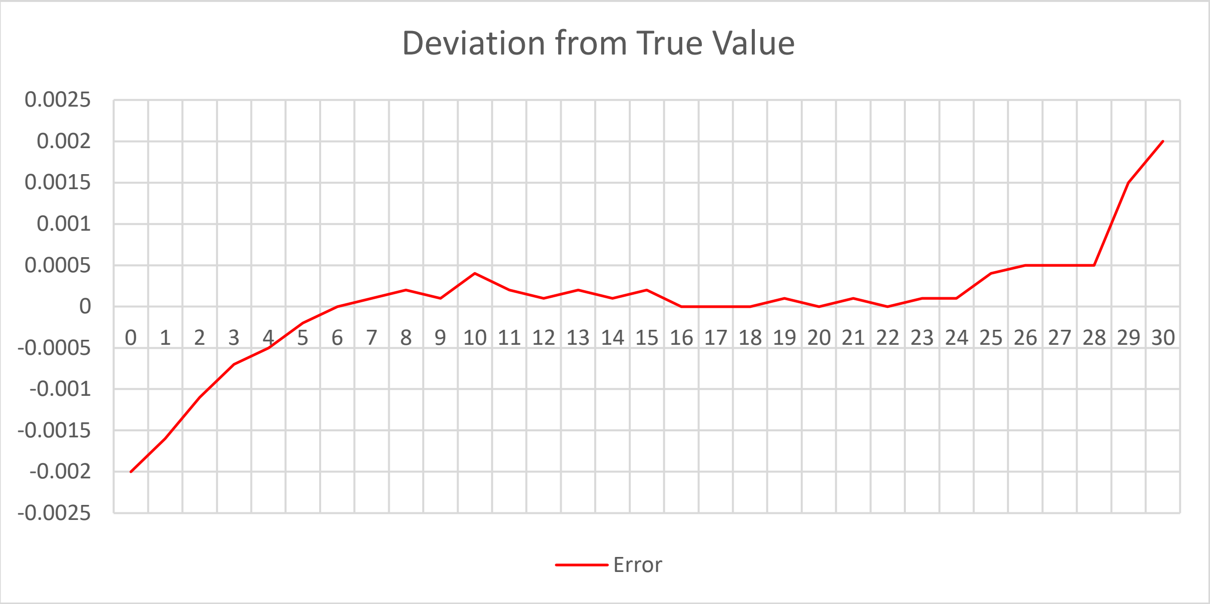
$$\boxed{y = -4 \times 10^{-5}x^2 + 0.018x - 0.0020} \qquad \text{... (9)}$$

ERROR IN THE NEW QUADRATIC EQUATION:

Now if we see the error and the comparison graphs between the actual value of sine wave and the value obtained from quadratics then,

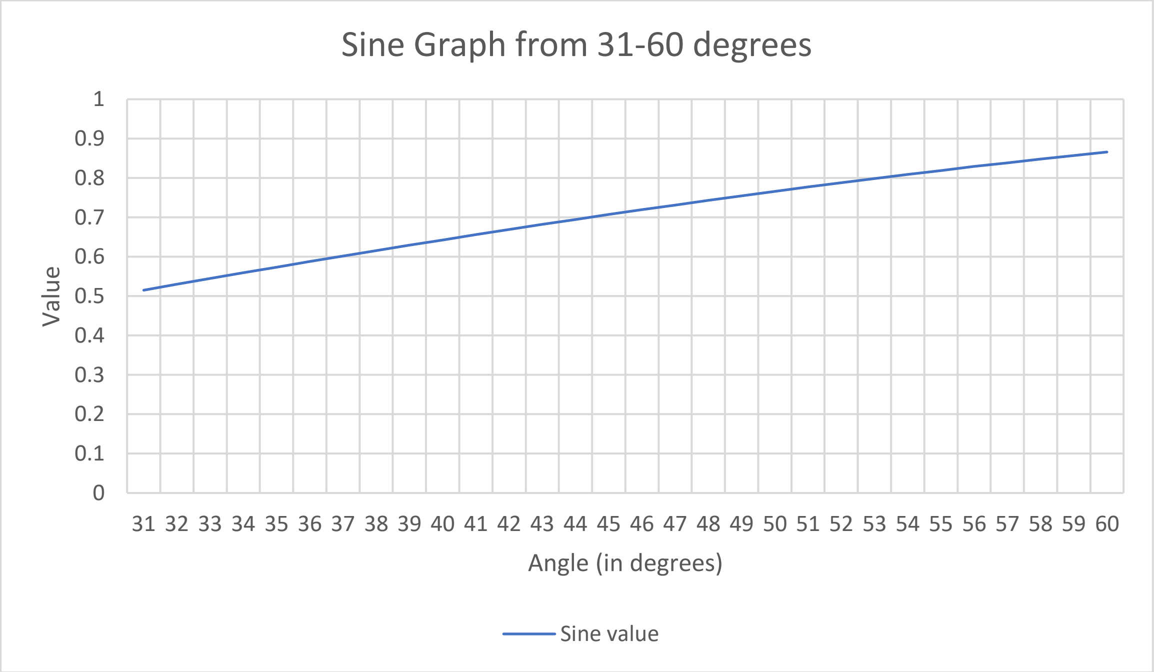


To see the errors, if we plot the errors only then we see that the error is significantly reduced!



This completes our first sub-range of quadratic approximation!

2. **Range from (31-60)**



Here also we will follow the same approach, but wait! This time it will be quite easy to figure out that which set of three angles will give almost close value for our quadratic.

If we see the pattern above, {20,40,60,80} gave correct approximate for cubic whereas {9,18,27} gave correct approximate for the first range of cubic equation. Now in both cases/sets the last term of the set is almost 11.11% behind the last term of our range, 90 in case of cubic and 30 in case of quadratic. The rest of the terms {20,40,60} or {9,18} follows our last term in AP.

So here also if we follow the same sequence then, our three angles to be chosen will be {39,48,57} as 57 is approx. 11.11% behind the last term of our sub-range i.e., 60 and the other two follows in AP.

Now, let the quadratic equation be $px^2 + qx + r = 0$, again the same process is followed. The equations used to get the values of constants 'a' and 'b' are,

$0.6293 = 1521p + 39q + r$ ... (10)
$0.7431 = 2304p + 48q + r$ ... (11)
$0.8387 = 3249p + 57q + r$ ... (12)

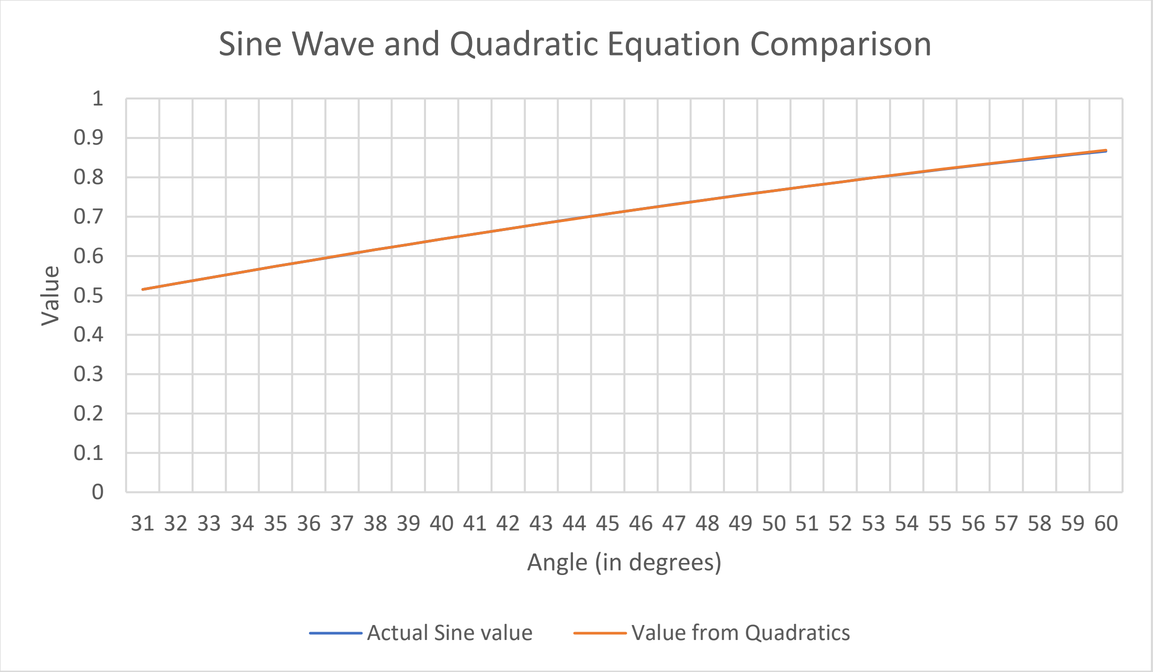
On solving these equations, we get the value of 'p' as $-1.1 \times 10^{-4} \approx -1 \times 10^{-4} \equiv -0.0001$ and the value of 'q' as 0.0213.

Before going to find the value of 'r' using errors, if we take r=0 and if we find the value of 31 which comes out to be 0.5642 and if we subtract 0.5150 (true sine value of 31 degrees) then it comes out to be '0.0492' now negative of this will be our tentative value of 'r'

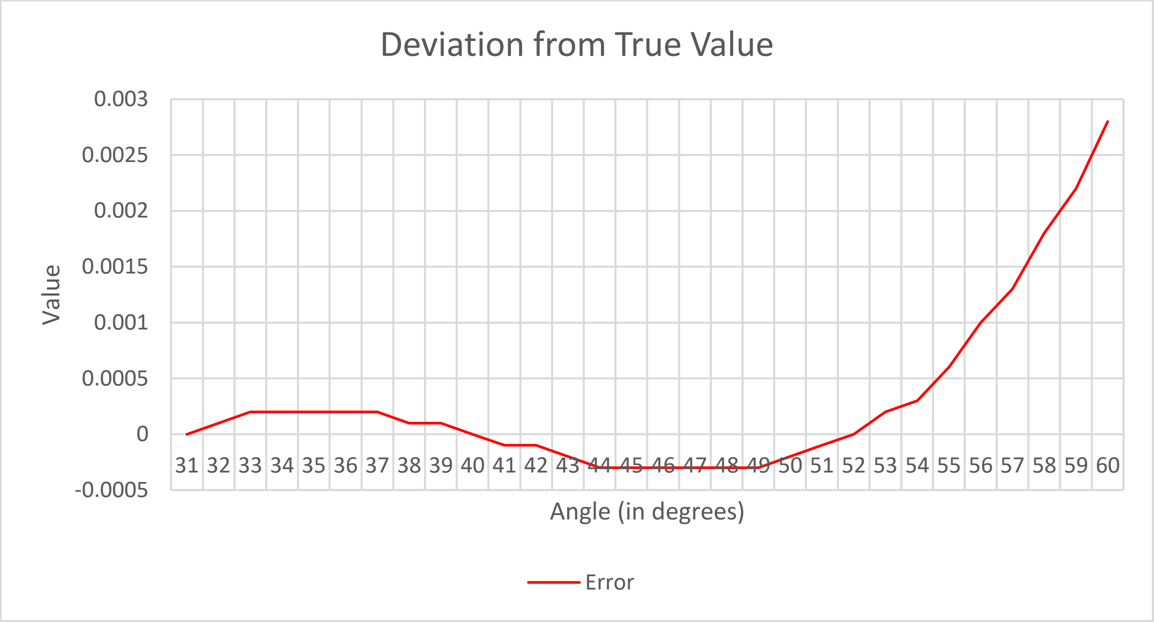
Hence our tentative quadratic looks like $y = -1 \times 10^{-4}x^2 + 0.0213x - 0.0492$ ... (13)

GRAPH AND IT'S DEVIATION FROM ACTUAL VALUE

If we graph the actual values of the sine function (up to 4 decimal places) and the values obtained by quadratic equation ($y = -1 \times 10^{-4}x^2 + 0.0213x - 0.0492$) (up to 4 decimal places) then the graph will look something like this.



As we see that both the graphs almost overlap each other but there are errors in both the values. If we graph the errors then,



Since the maximum value of error occurs at 60 which is equal to +0.0028. Here also we will use the method of averages as the quadratic equation cannot be formed with the given graph of errors and if we go for a higher degree of polynomial then main motive to find the quadratic equation will be affected.

So, by the method of averages, we get: $\frac{\Sigma\, Errors}{30} = \frac{0.009}{30} = 0.0003.$

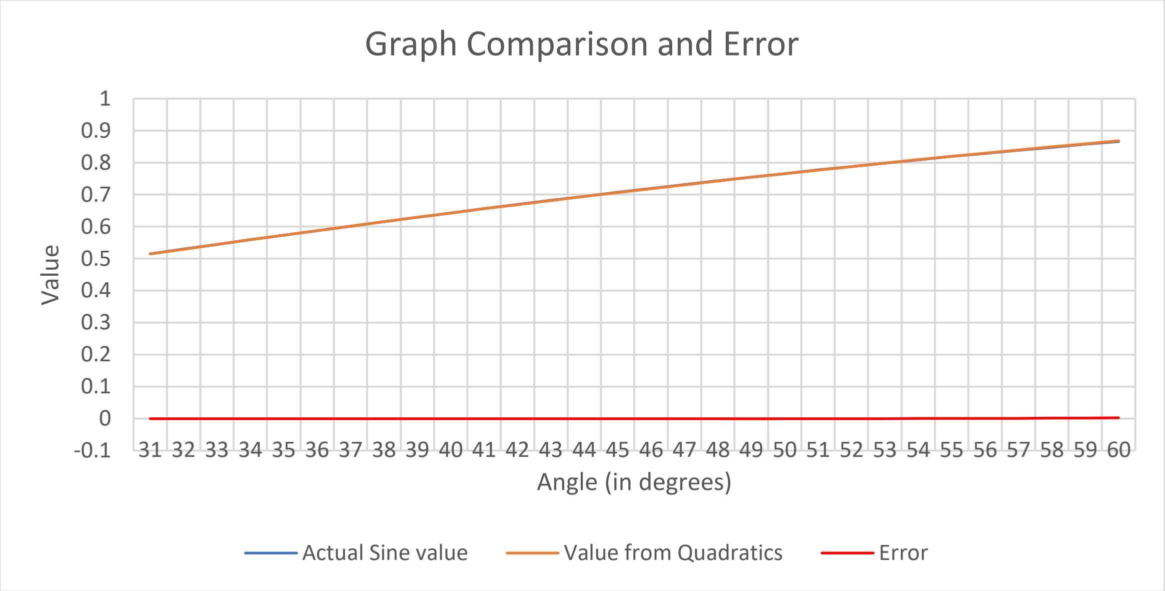
Hence the value of the average of the errors is 0.0003 which is to be subtracted from the tentative quadratic equation to get the final quadratic equation which will significantly reduce the error.

Hence our final equation for the second sub-range (31-60) will be:

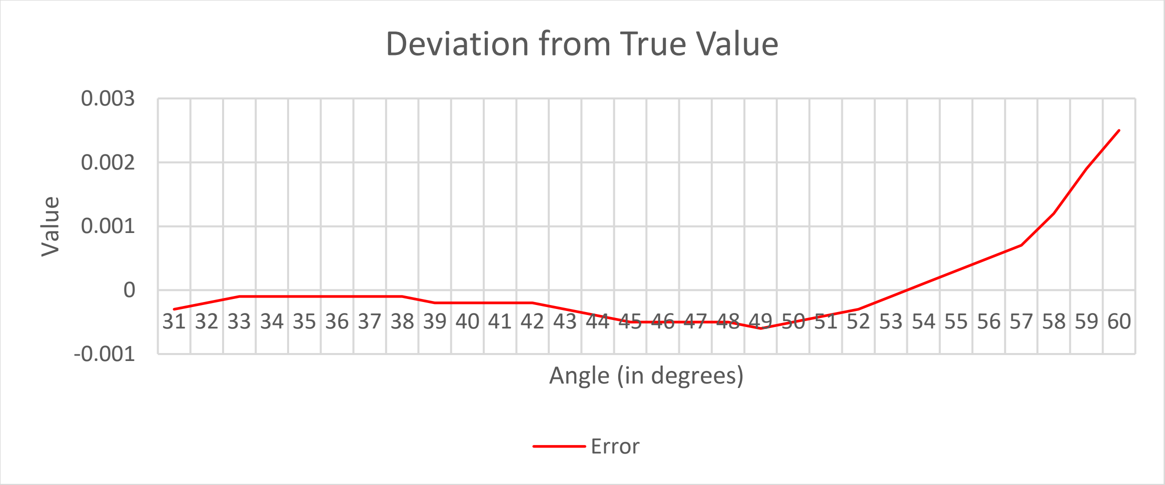
$$\boxed{y = -1 \times 10^{-4}x^2 + 0.0213x - 0.0495} \qquad \text{... (14)}$$

ERROR IN THE NEW QUADRATIC EQUATION:

Now if we see the error and the comparison graphs between the actual value of sine wave and the value obtained from quadratics then,

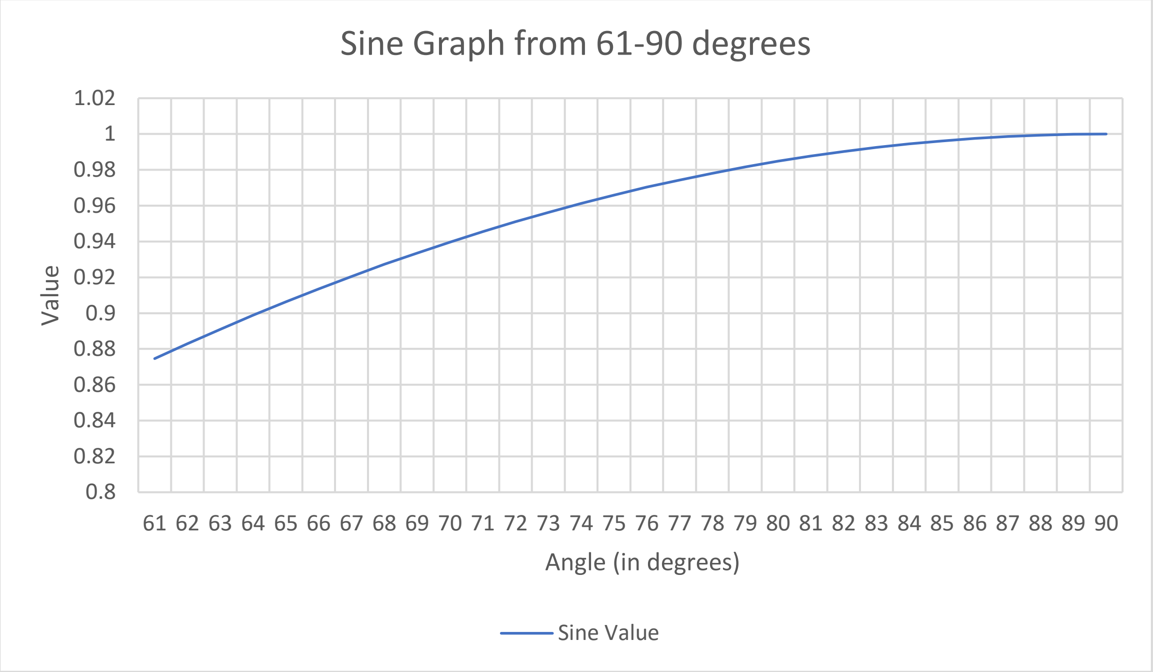


If we plot the errors, we see that the error is significantly reduced!



This completes our second sub-range of quadratic approximation!

Range from (61-90):



Here, in this case, the pattern did not work as seen above, I.e. {69,78,87} did not work to get the approximation probably because of the shape of the curve. So, after trying for different sets of triplets such as {61,75,89}, {72,82,92}, {65,75,85}, finally {63,72,81} works fine.

But wait! If we observe the set {63,72,81} then this triplet also follows the same pattern but in reverse order! 63 (first term) is approximately 11.11% ahead of 60 and the other two follows 63 in AP with common difference 9. This set is also justified as these three points play a major role in bending of the curve.

So now we have our set of angles required to form three equations which are:

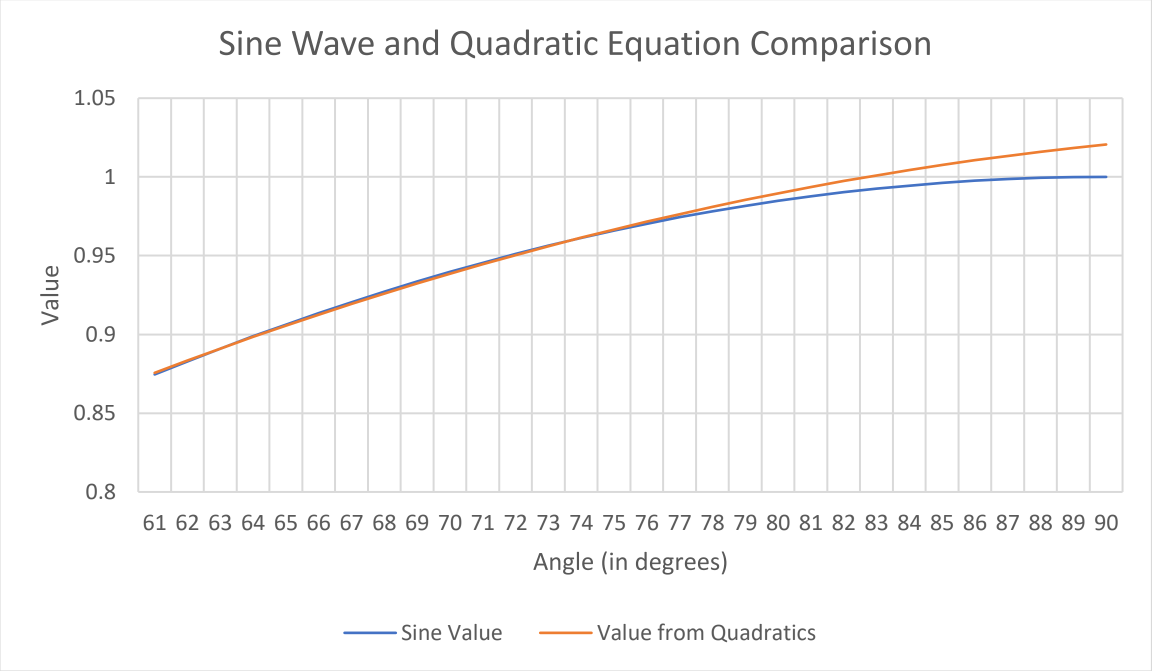
$$0.8910 = 3969l + 63m + n \quad \text{... (15)}$$
$$0.9511 = 5184l + 72m + n \quad \text{... (16)}$$
$$0.9877 = 6561l + 81m + n \quad \text{... (17)}$$

On solving these equations, we get the value of ‘l’ as $-1.4 \times 10^{-4} \approx -1 \times 10^{-4}$, the value of ‘m’ as 0.0201 and the value of ‘n’ as 0.0216. The value of ‘n’ is tentative and can be changed to minimize the error. Hence our tentative quadratic equation looks like,

$$y = -0.0001x^2 + 0.0201x + 0.0216 \quad \text{... (18)}$$

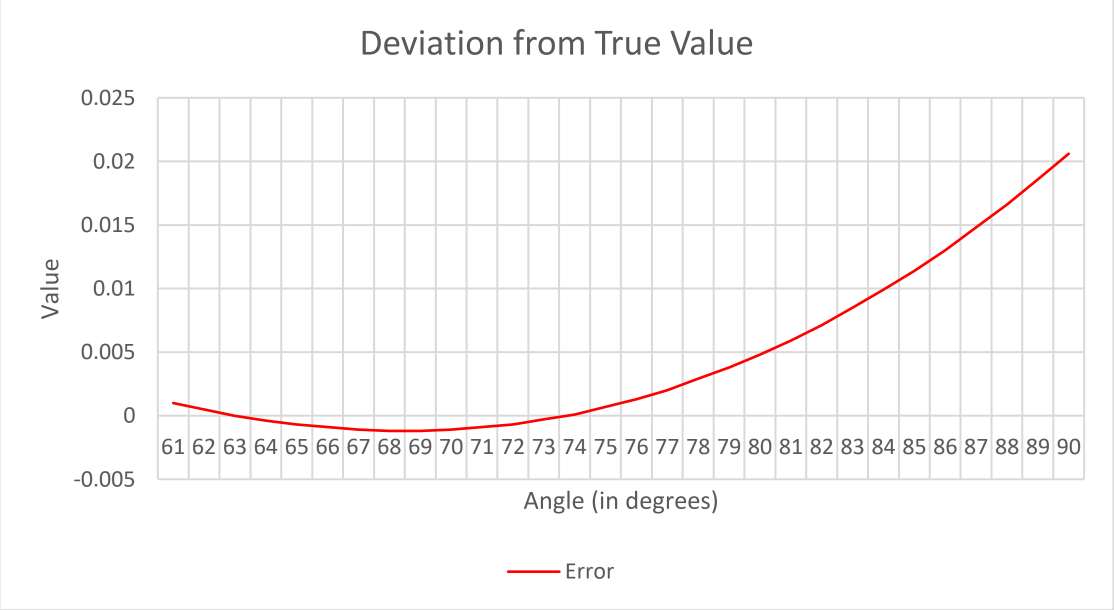
GRAPH AND IT'S DEVIATION FROM THE ACTUAL VALUE.

If we graph the actual values of the sine function (up to 4 decimal places) and the values obtained by quadratic equation ($y = -0.0001x^2 + 0.0201x + 0.0216$) (up to 4 decimal places) then the graph will look something like this.



The error from the actual sine wave and that through quadratics can be easily seen.

If we graph the error then,



So, by the method of averages, we get: $\frac{\Sigma \, Errors}{30} = \frac{0.135}{30} = 0.0045$

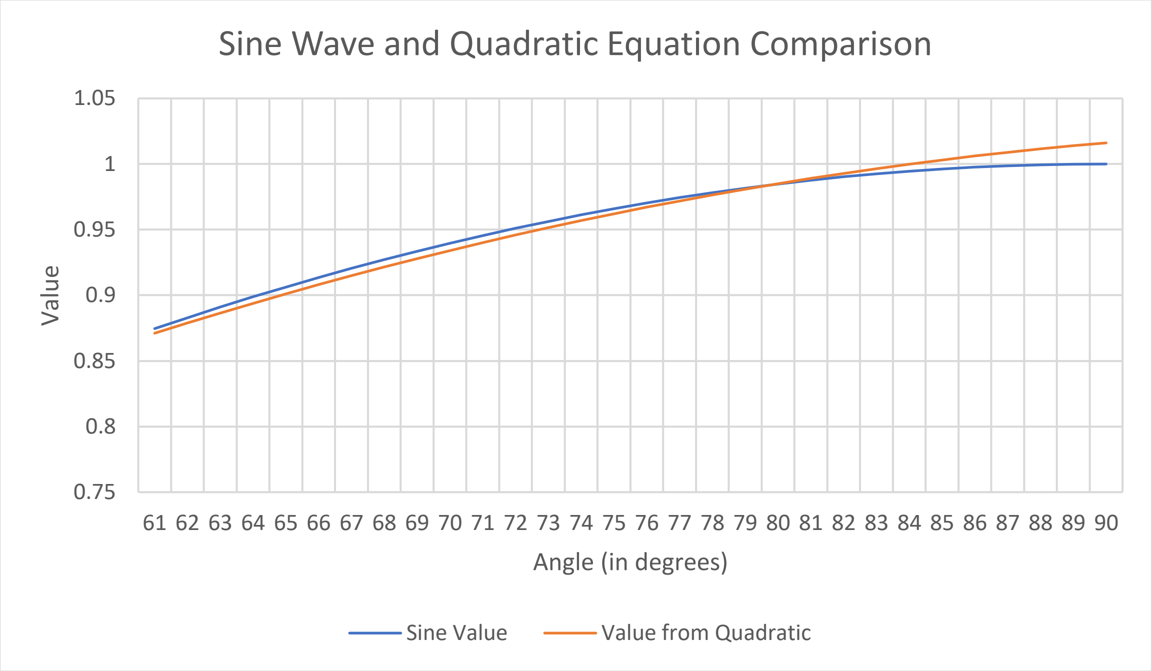
Hence the value of the average of the errors is 0.0003 which is to be subtracted from the tentative quadratic equation to get the final quadratic equation which will reduce the error to some extent.

Hence our new equation looks like:

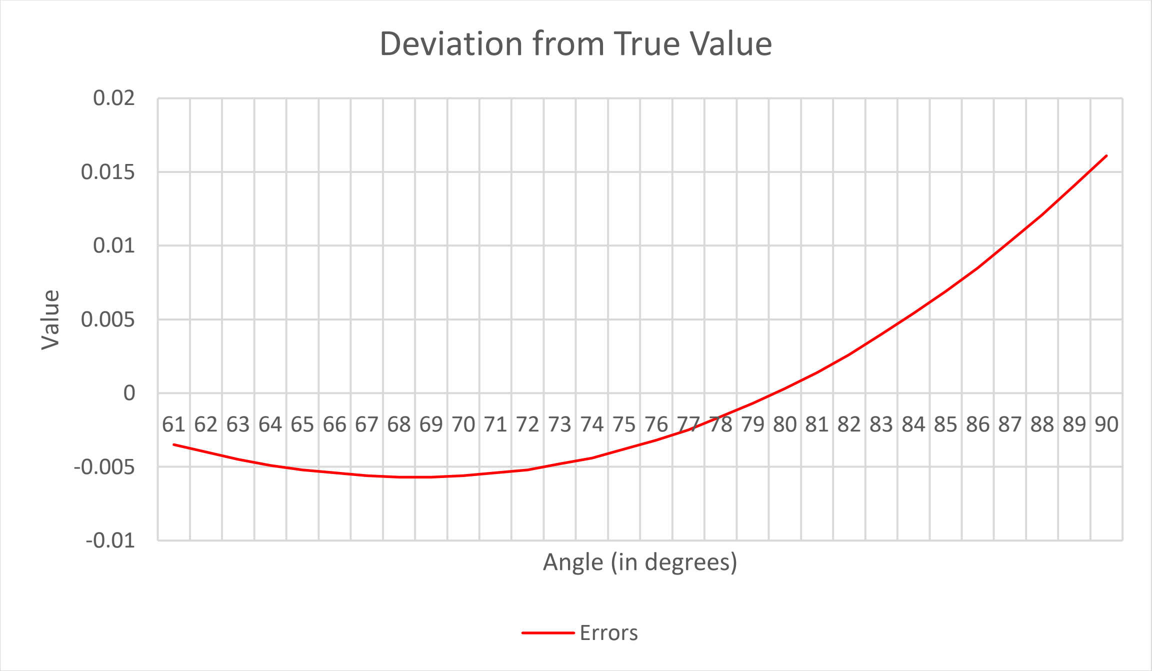
$$y = -1 \times 10^{-4}x^2 + 0.0201x + 0.0171 \qquad \ldots (19)$$

ERROR IN THE NEW EQUATION:

Now if we see the comparison graphs between the actual value of sine wave and the value obtained from quadratics then,



Now if we graph the error only then,



But wait! If we observe the graph from the error then we see that the quadratic equation of that curve can be easily formed. And then the equation of that error will be subtracted from our original to get even the closest approximate.

Now to form the quadratic equation we follow the same procedure (I have shifted the numbers from (61-90) to (1-15) for time being and then revert to the original situation) which will get us to the following equation:

$$y = 5 \times 10^{-5} x^2 - 0.0008x + k \qquad \text{... (20)}$$

Here ‘k’ is unknown, to find it first we substitute k=0 and with this substitution, we get the value of y at x=15 as -0.00075 and to make it equal to the error at x=15 i.e., -0.0038 we subtract these we get the value of ‘k’ as -0.00305.

Hence our new equation becomes: $y = 5 \times 10^{-5}x^2 - 0.0008x - 0.00305$ ... (21)

Now this equation is valid for numbers between 1-30 to get it in the range of (61-90) we subtract 60 as follows:

$$y = 5 \times 10^{-5}(x - 60)^2 - 0.0008(x - 60) - 0.00305 \quad \text{... (22)}$$

Hence on solving this equation we get the final equation for the error as:

$$\boxed{y = 5 \times 10^{-5}x^2 - 0.0068x + 0.22495} \quad \text{... (23)}$$

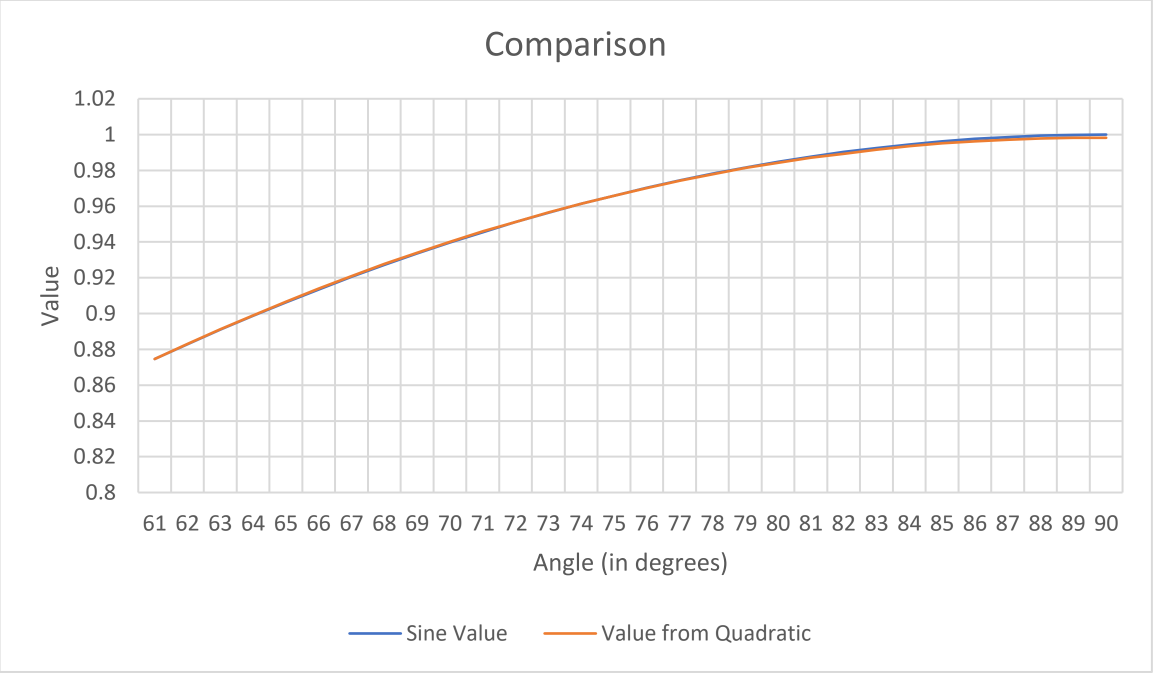
To get the final quadratic equation of our third sub-range, we subtract the equation (23) from (19).

Hence the final equation for the third sub-range is:

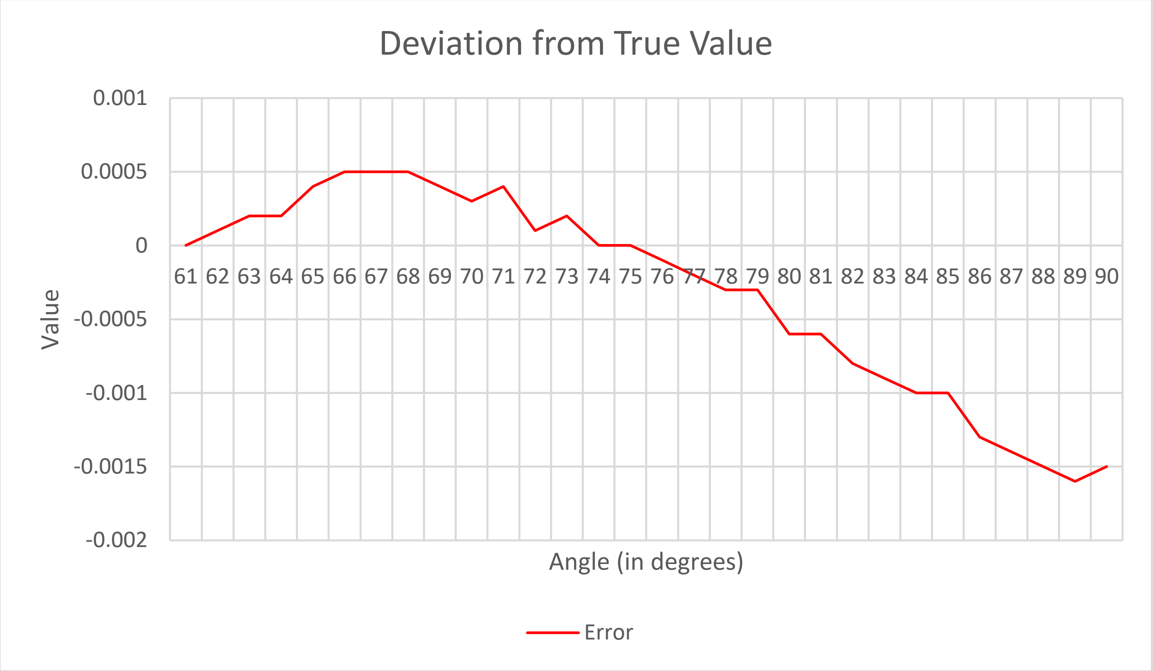
$$\boxed{y = -1.5 \times 10^{-4}x^2 + 0.0269x - 0.20785} \quad \text{... (24)}$$

TESTING THE FINAL EQUATION

Now if we see the comparison graphs between the actual value of sine wave and the value obtained from quadratics then,



If we plot the errors, we see that the error is significantly reduced!



This completes our third and final sub-range of the quadratic equation!

RESULTS AND CONCLUSION:

1. Cubic Equation:

The cubic equation formed which is used for the approximation of the sine function is:

$$\boxed{y = -6 \times 10^{-7}x^3 - 2 \times 10^{-5}x^2 + 0.018x}$$

2. Quadratic Equation:

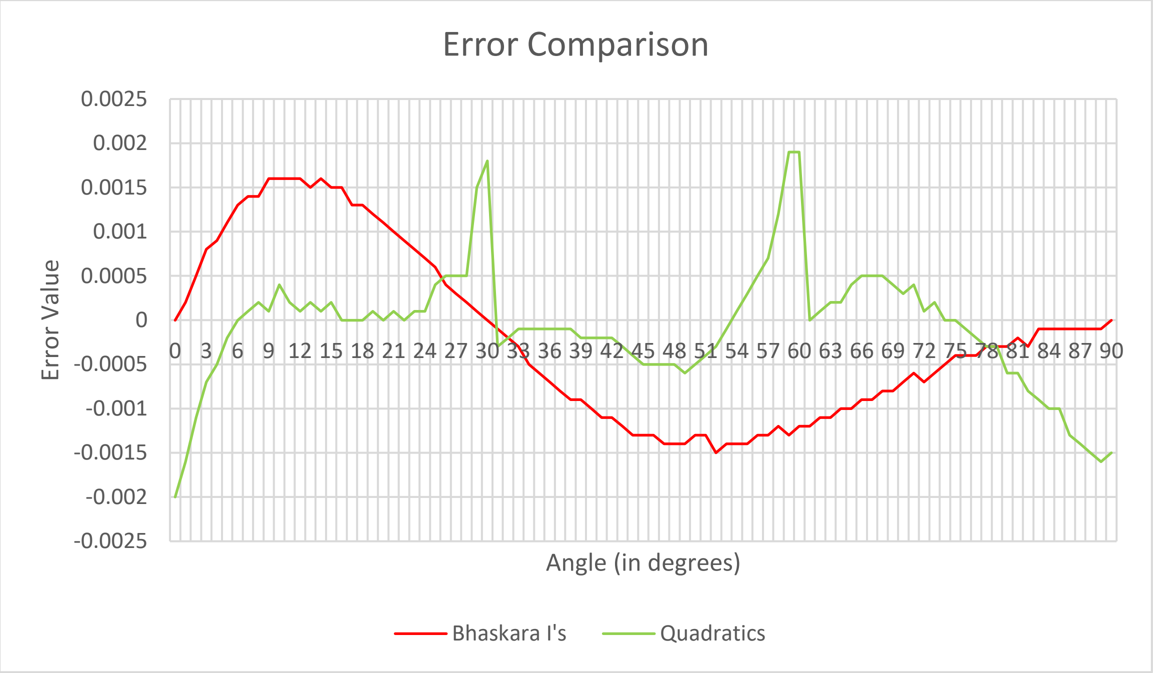
The quadratic equations formed which are used for the approximation of the sine function are:

$$\boxed{\begin{aligned} &y = -4 \times 10^{-5}x^2 + 0.018x - 0.0020 &&; \{for\ x \in (0° - 30°)\} \\ &y = -1 \times 10^{-4}x^2 + 0.0213x - 0.0495 &&; \{for\ x \in (31° - 60°)\} \\ &y = -1.5 \times 10^{-4}x^2 + 0.0269x - 0.20785 &&; \{for\ x \in (60° - 90°)\} \end{aligned}}$$

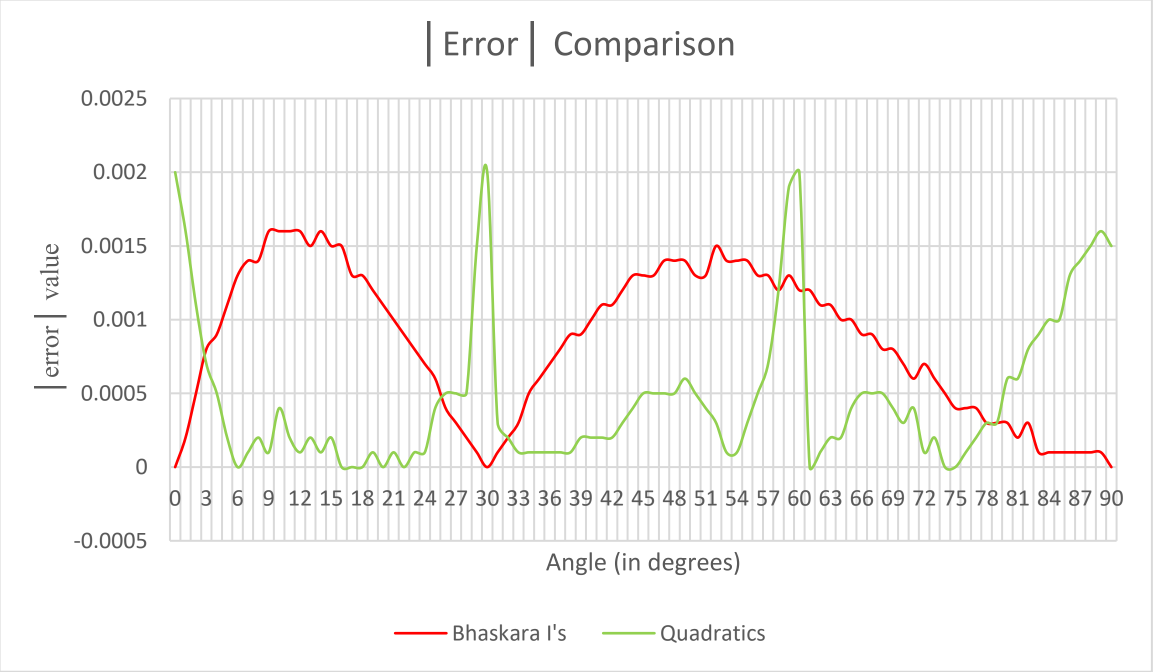
Hence the equations formed above provide the good approximation for the sine function. In this, I try to find the quadratic equation with three different ways to find the constant term of that quadratic equation. For most angles the error is even less than the error by Bhaskara I's sine approximation formula. **All the equations are derived from my limited knowledge of high school**.

A CONTRAST COMPARISON BETWEEN BHASKARA I's SINE APPROXIMATION FORMULA AND QUADRATIC EQUATIONS

In order to compare Bhaskara I's formula with my quadratic equations on the basis of errors, we plot the errors of Bhaskara I's formula and Quadratics on the same graph then,



To if we plot the $|errors|$ of the same, in order to visualise the data more clearly then,



OBSERVATION:

1. For most of the angles the error obtained in the case of quadratics is much smaller than in the case of Bhaskara I's sine approximation formula.
2. The error in case of quadratics increases only when the range changes, i.e., at 0,30,60 and 90 degrees
3. The Bhaskara I gave only one single equation for the computation of the sine values whereas in case of quadratics, I have formed 3 sets of different quadratic equation for the computation of the sine values.

REFERENCES:

1. I would like to mention 'desmos;(https://www.desmos.com/calculator)' an open-source online graphing calculator for testing the equations that I made.
2. Just to know the previous work on this topic I have visited the following sites:
   (a) Bhaskara I's sine approximation formula - Wikipedia.
   (b) Taylor's Series of sin x (mit.edu)